<u>Quantities vs. Amounts vs. Units</u>

For related rates and optimization problems,
you must be able to distinguish between a quantity, versus an amount, versus a unit.

A quantity is a property of an object (typically) that can be assigned a numerical value.
A quantity should <u>NOT</u> usually include any numbers or units.

eg. weight of a person, length of a beam, price of a car are quantities
person, beam, car are **<u>NOT</u>** quantities (they are objects)
pounds, feet, dollars are **<u>NOT</u>** quantities (they are units)

An amount is a number that represents a measurement of a quantity.

A unit is the scale given to an amount.

eg. pounds, feet, hours, dollars are units

Examples

If a person is 6 feet tall
the quantity is the height of the person
the amount is 6 and the unit is feet OR
the amount is 72 and the unit is inches OR
the amount is 2 and the unit is yards

If a ball is falling at 30 feet per second
the quantity is the speed of the ball
the amount is 30 and the unit is feet per second OR
the amount is 20 5/11 and the unit is miles per hour

Notice that the amount and unit in a single situation can have multiple (interdependent) values,
but the quantity remains the same.

What are the quantities, amounts and units in the following situations ?
Find another amount and unit pair that is equivalent to the ones mentioned.

[a] A bag of rice is 50 pounds.

[b] A storage shed holds 400 cubic feet.

[c] A diver descends 25 meters.

[d] The movie lasted 2 hours.

[e] My aunt just turned 50.

[f] The ferris wheel goes around once every 3 minutes.

[g] The Tropic of Cancer is located at 23° north.

[h] LA is 347 miles from SF.

[i] A haircut is $15.

[j] A ladder is 10 feet.

[k] A piece of cloth is 2 square feet.

[l] The movie started at 7pm.

[m] The engine is turning at 3000 rpm.

[n] 3 square feet of wrapping paper are needed to wrap a gift box.

[o] The odometer reads 35000 kilometers.

ANSWERS For QUANTITIES vs AMOUNTS vs UNITS

[a] The quantity is the weight of the rice bag.
The amount is 50 and the unit is pounds OR
the amount is 22 8/11 and the unit is kilograms.

[b] The quantity is the volume of the storage shed.
The amount is 400 and the unit is cubic feet OR
the amount is 11,327 and the unit is liters.

[c] The quantity is the depth of the diver.
The amount is 25 and the unit is meters OR
the amount is 82 and the unit is feet.

[d] The quantity is the duration of the movie.
The amount is 2 and the unit is hours OR
the amount is 120 and the unit is minutes.

[e] The quantity is the age of my aunt.
The amount is 50 and the unit is years OR
the amount is 5 and the unit is decades.

[f] The quantity is the period of the ferris wheel.
The amount is 3 and the unit is minutes OR
the amount is 180 and the unit is seconds.

[g] The quantity is the latitude of the Tropic of Cancer.
The amount is 23 and the unit is degrees.

[h] The quantity is the distance between LA and SF.
The amount is 347 and the unit is miles OR
the amount is 555 and the unit is kilometers.

[i] The quantity is the price of the haircut.
The amount is 15 and the unit is dollars OR
the amount is 1500 and the unit is cents.

[j] The quantity is the length of the ladder.
The amount is 10 and the unit is feet OR
the amount is 120 and the unit is inches.

[k] The quantity is the area of the cloth.
The amount is 2 and the unit is square feet OR
the amount is 288 and the unit is square inches.

[l] The quantity is the starting time of the movie.
The amount is 7 and the unit is hours since noon OR
the amount is 19 and the unit is hours since midnight.

[m] The quantity is the angular (rotational) speed of the engine.
The amount is 3000 and the unit is revolutions per minute OR
the amount is 6000π and the unit is radians per minute.

[n] The quantity is the surface area of the gift box.
The amount is 3 and the unit is square feet OR
the amount is 432 and the unit is square inches.

[o] The quantity is the total distance travelled by the vehicle.
The amount is 35000 and the unit is kilometers OR
the amount is 21875 and the unit is miles.

Related Rates Problem Solving Procedure (Section 3.9)

[1] Draw a diagram, if possible.
Do not label any numbers on the diagram, unless the quantities they represent NEVER change.

[2] Identify any **quantity/quantities** whose rate of change information is/are given or can be calculated directly by differentiation or other means.

[3] Identify the **quantity** whose rate of change you want, and under what circumstances.

[4] Find an equation connecting the quantities in [2] and [3].

[5] Differentiate [4] implicitly with respect to time.

[6] Substitute all known information (you may need to use [4] or other equations).

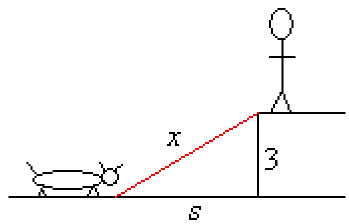
[7] Solve for the desired rate of change. Sanity check the sign and units of the answer.

Related Rates Example Problem using Procedure

You are standing on a table 3 feet off the ground. A rat is running on the ground directly towards the table, so that its distance from you is decreasing by 6 inches per second. How quickly is the rat moving when it is 4 feet from you ?

A complete solution only needs to include the work shown in black.

[1] Draw a diagram, if possible.
Do not label any numbers on the diagram, unless the quantities they represent NEVER change.



[2] Identify any quantity/quantities whose rate of change information is/are given or can be calculated directly by differentiation or other means.

If quantity x = distance between you and rat, we are given that $\frac{dx}{dt} = -0.5$ ft/s

[3] Identify the quantity whose rate of change you want, and under what circumstances.

If quantity s = distance between rat and table, we want $\left.\frac{ds}{dt}\right|_{x=4\ ft}$

[4] Find an equation connecting the quantities in [2] and [3].

The equation connecting x and s is $s^2 + (3\ ft)^2 = x^2$

[5] Differentiate [4] implicitly with respect to time.

$$\frac{d}{dt}[s^2 + 9\ ft^2] = \frac{d}{dt}[x^2]$$
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

[6] Substitute all known information (you may need to use [4] or other equations).

$$2(\sqrt{7}\ ft)\left.\frac{ds}{dt}\right|_{x=4\ ft} = 2(4\ ft)(-0.5\ \frac{ft}{s})$$

NOTE: $s^2 + 9\ ft^2 = (4\ ft)^2 \Rightarrow s = \sqrt{7}\ ft$

[7] Solve for the desired rate of change.

$$\left.\frac{ds}{dt}\right|_{x=4\ ft} = -\frac{2}{\sqrt{7}}\ \frac{ft}{s}$$

Sanity check: the distance between the rat and the table is decreasing as the rat runs towards the table, so the derivative is negative. Also, s is in feet and t is in seconds, so derivative is in feet per second.

To graph a function *f(x)* from its formula (Section 4.5):

[1] Find the domain and discontinuities of $f(x)$.

[2] Find the *y*-intercepts (set $x=0$) and *x*-intercepts (solve $f(x)=0$) (if possible)

[3] At each discontinuity $x=a$ of $f(x)$, find $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ to determine whether there is a removable discontinuity, a jump discontinuity, an infinite discontinuity (a vertical asymptote), or none of the above, at *a*.

Removable discontinuity: $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ and both limits exist
Jump discontinuity: $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$ and both limits exist
Infinite discontinuity: $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$

[4] Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine if there are any horizontal asymptotes.

[5] Solve $f'(x)=0$ or undefined, and $f''(x)=0$ or undefined within the domain, and find the values of $f(x)$ at those points.

[6] For each point in [5] where $f'(x)$ is undefined and each removable and jump discontinuity in [3], determine if $\lim_{x\to a^+} f'(x) = \pm\infty$ or $\lim_{x\to a^-} f'(x) = \pm\infty$ to find vertical tangent lines.

[7] Draw a number line subdivided into intervals by the points in [3] and [5],
and mark all discontinuities in [3] by type,
all horizontal tangent lines in [5] ($f'(x)=0$)
and all vertical tangent lines in [6].

[8] Find the signs of $f'(x)$ and $f''(x)$ within each interval.

[9] Analyze the sign changes of $f'(x)$ and $f''(x)$ between each pair of intervals, and mark all inflection points and all local extrema by type.

[10] Analyze the signs of $f'(x)$ and $f''(x)$ within each interval, and sketch a representative shape based on concavity and direction (increasing/decreasing).

[11] Plot the asymptotes in [3] and [4],
the limit points in [3],
the points in [2] and [5],
and connect using the shapes in [10].

SUMMARY: find domain, discontinuities, intercepts, asymptotes
number line = discontinuities, $f'(x)$ *or* $f''(x)=0$ or undefined
perform sign analysis on $f'(x)$ and $f''(x)$ and sketch segments
find horizontal/vertical tangent lines, local max/min, inflection points
connect

<u>Optimization Problem Solving Procedure (Section 4.7)</u>

[1] Draw a diagram, if possible.

[2] Name the **<u>quantity</u>** you want the largest/maximum or smallest/minimum value of.

[3] Name the **<u>quantity/quantities</u>** you can change to try to get the maximum or minimum value of [2].

[4] Find the formula for the quantity in [2] in terms of the quantity/quantities in [3].

IF THE FORMULA IN [4] INVOLVES MORE THAN ONE VARIABLE FROM [3].

> [4a] Find any restrictions/relationships between the quantities in [3] that do not involve any other quantities.
>
> [4b] Solve [4a] in terms of one quantity.
>
> [4c] Rewrite [4] in terms of one quantity using [4b].

[5] Find the smallest and largest allowable values of the independent quantity in [4].

[6] Use calculus to find the global maximum/minimum of the dependent quantity in [4] over the domain in [5].